MATH6031 Lecture 9

SYZ mirror symmetry w/ corrections I

(X, D) $\quad$ X : Kähler

D : effective anticanonical divisor w/ simple normal crossings

Rmk : The complement $X \setminus D$ is called a log Calabi-Yau variety.

(X, D) is mirror to the Landau-Ginzburg (LG) model

$$(\check{X}, W)$$

s.t. • $\check{X}$ is the semi-flat mirror to $X \setminus D$:

$$\begin{array}{ccccc} X \supset X\setminus D = T^*B_0/\check{\Lambda} & & \check{X} := TB_0/\Lambda \\ \pi \searrow \quad \pi \searrow & & \swarrow \\ \{(L,\nabla)\}/\sim \qquad B \supset B_0 = B \setminus \partial B & & \end{array}$$

• $W : \check{X} \to \mathbb{C}$ is the Lagrangian Floer potential (where $\check{X} = \{(L,\nabla)\}/\sim$) defined by

$$m_0(L,\nabla) = W(L,\nabla)\cdot[L]$$

(m_0: obstruction chain for (L,∇) in X)

Toric examples

Consider a toric Kähler manifold

$$(X = X_\Sigma = X_\Delta, \omega, J)$$

(Σ: fan, Δ: polytope)

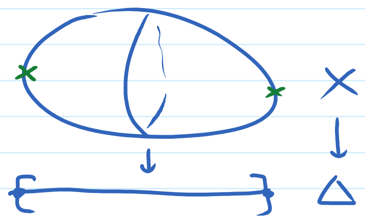
The moment map

$$\begin{array}{ccc} T^n \curvearrowright X^{n_{\mathbb{C}}} & \supset & D = \text{toric divisor} = \bigcup_{i=1}^m D_i \\ \pi \downarrow & & \downarrow \\ \Delta & \supset & \partial\Delta \end{array}$$

(D_i: prime divisor)

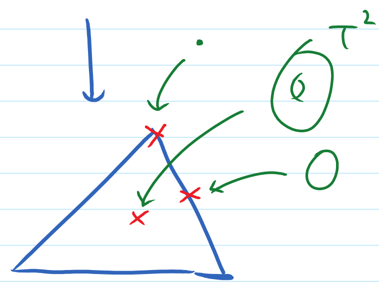
is a Lagrangian torus fibration, w/ fibers collapsing is a Lagrangian torus fibration, w/ fibers collapsing to lower dim'l tori over $\partial\Delta$ and $X \setminus D \to \Delta^\circ := \Delta \setminus \partial\Delta$ is a Lagrangian torus fibration w/o sing. fibers.

e.g. • $X = \mathbb{P}^1$, $\Delta = [-1, 1]$



Rmk $\Longrightarrow$ $\chi(\mathbb{P}^1) = 2$

• $X = \mathbb{P}^2$, $\Delta =$ triangle



Rmk $\Longrightarrow$ $\chi(\mathbb{P}^2) = 3$

We study mirror symmetry for (X, D).

$$\begin{array}{ccccc} D & \subset & X & \supset & X \setminus D =: X_\circ \cong (\mathbb{C}^\times)^n \text{ (holomorphically)} \\ \downarrow & & \downarrow \pi & & \downarrow \pi \\ \partial\Delta & \subset & \Delta & \supset & \Delta^\circ \end{array}$$

We have $\Delta \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ where $M \cong \mathbb{Z}^n$ is a lattice, $M = \operatorname{Hom}((\mathbb{C}^\times)^n, \mathbb{C}^\times)$ character group ($\curvearrowleft X$)

and $X_\circ := \pi^{-1}(\Delta^\circ) \cong T^*\Delta^\circ / N$

where $N := M^\vee = \operatorname{Hom}(M, \mathbb{Z})$.

$\Rightarrow$ $\check{X} := T\Delta^\circ / M \subset TM_{\mathbb{R}}/M \overset{\text{can}}{=} (\mathbb{C}^\times)^n$ $\quad$ $\mathbb{R}^n = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$, $T\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{C}^\times)^n$

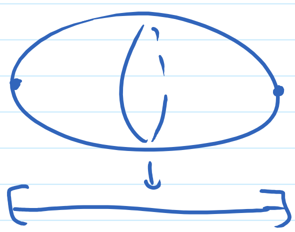
$= \operatorname{Log}^{-1}(\Delta^\circ)$ ← bounded domain in $(\mathbb{C}^\times)^n$

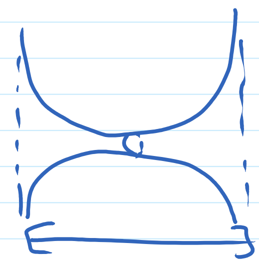
$= \mathrm{Log}(\Delta)$ ...

where $\mathrm{Log} : (\mathbb{C}^\times)^n \longrightarrow \mathbb{R}^n$, $(z_1, \dots, z_n) \longmapsto (\log|z_1|, \dots, \log|z_n|)$

$$\begin{array}{ccc} X \setminus D = X_0 & & \check{X} \subset TM_{\mathbb{R}}/M = (\mathbb{C}^\times)^n \\ & \searrow \quad \swarrow & \swarrow \mathrm{Log} \\ & \Delta^\circ \subset M_{\mathbb{R}} = \mathbb{R}^n & \end{array}$$

e.g. For $X = \mathbb{P}^1$

$X =$ 

$\check{X} =$ 

$W = z + \frac{q}{z}$

Thm (Cho–Oh 2005)

If X is Fano, i.e. $c_1(X) > 0$, then

$$n_\beta = \begin{cases} 1 & \text{if } \beta = \beta_i \text{ is basic} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad W = \sum_{i=1}^m Z_{\beta_i}$$

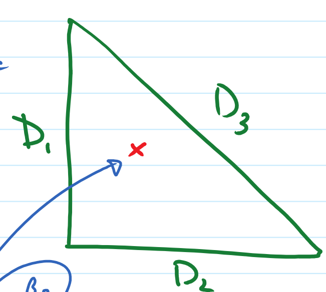
recall that

$$W(L, \nabla) = \sum_{\substack{\beta \in \pi_2(X, L) \\ \mu(\beta) = 2}} n_\beta \cdot Z_\beta(L, \nabla)$$

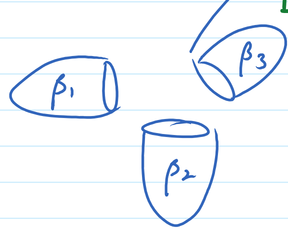
$Z_\beta(L,\nabla) = e^{-\int_\beta \omega} \cdot \mathrm{hol}_\nabla(\partial\beta)$

$$(ev_0)_*([\overline{\mathcal{M}}_1(L, \beta)]^{vir}) = n_\beta \cdot [L], \quad n_\beta \in \mathbb{Q}$$

Rmk: The result of Cho–Oh verifies a prediction by physicists (Hori–Vafa).

For $X = \mathbb{P}^2$, $\Delta =$ 

$\check{X} = T\Delta^\circ/M \subset (\mathbb{C}^\times)^2$, z_1, z_2



Cho–Oh $\Rightarrow$ $n_\beta = \begin{cases} 1 & \text{if } \beta \in \{\beta_1, \beta_2, \beta_3\} \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow \quad W = Z_{\beta_1} + Z_{\beta_2} + Z_{\beta_3} = z_1 + z_2 + \frac{q}{z_1 z_2}$$

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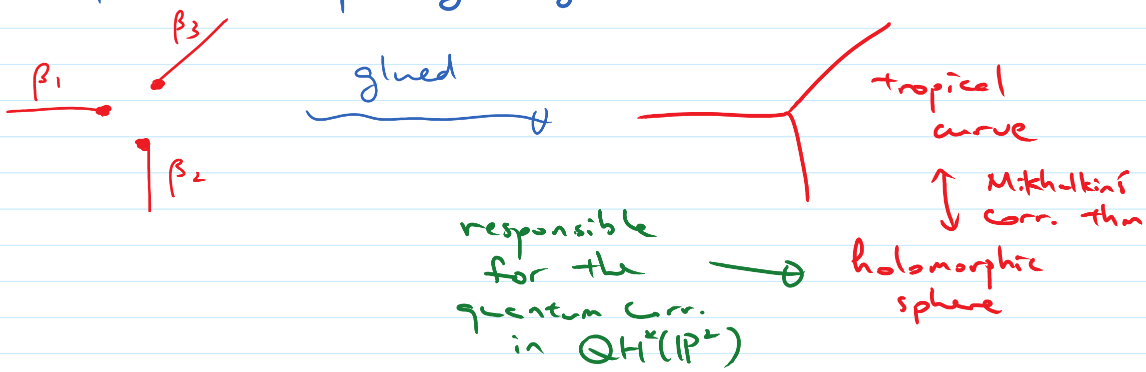
The isom.

$$QH^*(\mathbb{P}^2) \cong Jac(W)$$

can be explained tropically by



This can be generalized to big quantum cohomology (M. Gross 2009)

But if X is non-Fano, W has to be corrected by bubbled configurations which involves disks w/ higher Maslov indices and sphere components:

$$W = W_0 + \text{corr. terms}$$

$$W_0 = \sum_{i=1}^{m} Z_{\beta_i}$$

In particular, if X is semi-Fano, i.e. $c_1(X) \geq 0$, then only bubbled configurations with sphere bubbles will contribute

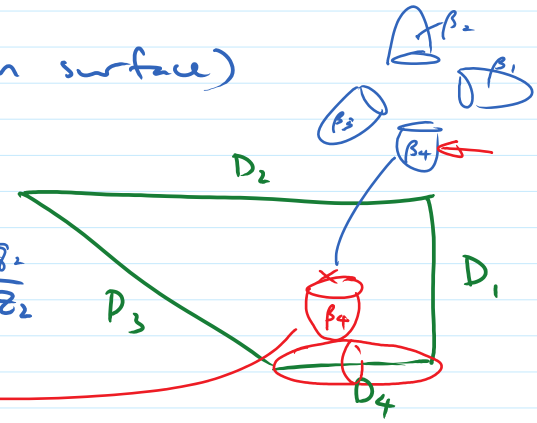
$$\Longrightarrow \; W = \sum_{i=1}^{m} (1 + \delta_i(q)) Z_{\beta_i} = W_0 + \text{corr. terms}$$

where $1 + \delta_i(q)$ is a generating function of genus 0, 1-pointed open Gromov-Witten invariants of (X, D).

1-pointed open Gromov-Witten invariants of (X, D).

e.g. For $X = \mathbb{F}_2$ (Hirzebruch surface)

$$= \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$$



$$W = z_1 + z_2 + \frac{q_1 q_2^2}{z_1 z_2^2} + (1 + q_1)\frac{q_2}{z_2}$$

$$= W_0 + \underbrace{q_1 \cdot \frac{q_2}{z_2}}_{\text{corr. term}}$$

(Auroux, FOOO, ...)

Rmk SYZ transform :

$$\mathcal{F}_{SYZ}\left(e^{i(\omega + \Psi)}\right) = e^W \Omega$$

SYZ w/ Corrections I

Setting : (X, D) X : Kähler

$D \subset X$: effective anticanonical divisor w/ snc.

s.t. ① $\exists$ a Lagr torus fibration

$$\begin{array}{ccc} D & \subset & X \\ \downarrow & & \downarrow \pi \\ \partial B & \subset & B \end{array}$$

possibly w/ singular fibers over $B \setminus \partial B$.

(i.e. B is an affine mfd w/ singularities and boundary)

② Maslov class $\mu(L_b)$ of smooth fibers over $B \setminus \partial B$ vanish.

Let $\Gamma \subset B \setminus \partial B$ be the discriminant locus.

$B_0 := (B \setminus \partial B) \setminus \Gamma$ $X_0 := X \setminus D$ ...

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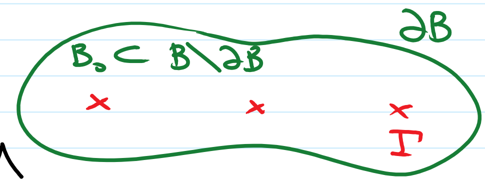
$$B_0 := (B\setminus\partial B)\setminus\Gamma \qquad\qquad X_0 \subset X\setminus D \supset D$$

$$X_0 := \pi^{-1}(B_0) \qquad\qquad \downarrow\pi \qquad \downarrow$$



Then we again have

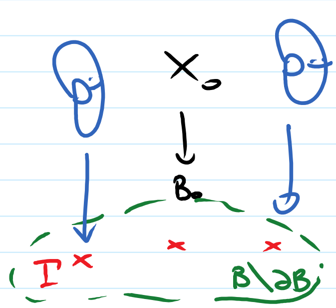
$$T^*B_0/\Lambda^\vee = X_0 \qquad\qquad \check{X}_0 := TB_0/\Lambda$$

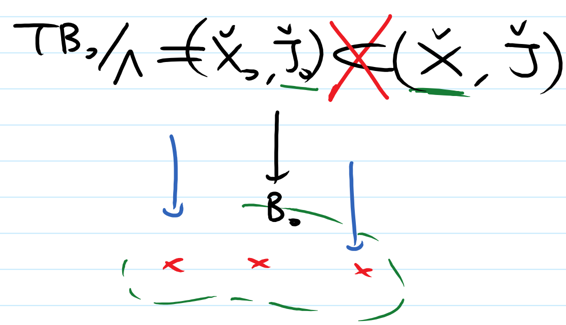
$$\pi \searrow \quad B_0 \quad \swarrow$$

Lagr. torus fibration w/o sing fibers $\to \pi$

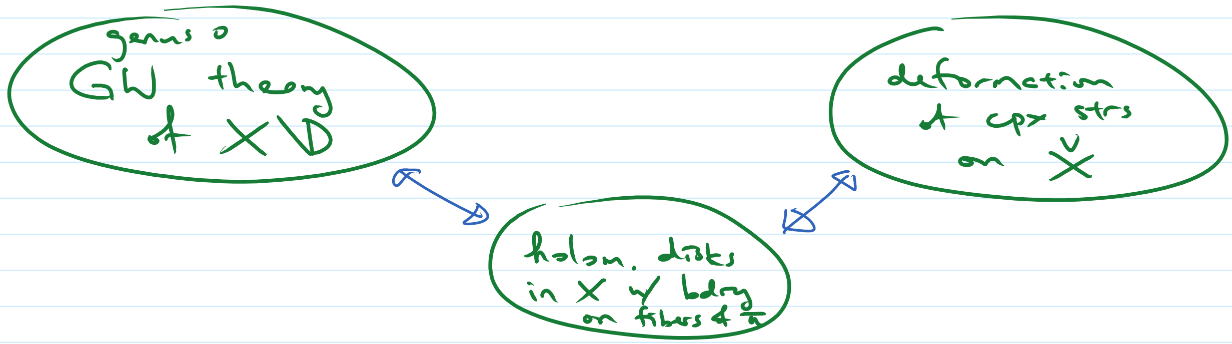
The issue is that, since we've deleted the singular fibers in $X\setminus D$, we shall (partially) compactify $\check{X}_0$ to get the correct mirror manifold.

Problem: The natural complex structure $\check{J}_0$ on $\check{X}_0$ cannot be extended to any partial compactification of $\check{X}_0$ because monodromy of the affine str. on B_0 is nontrivial around Γ.

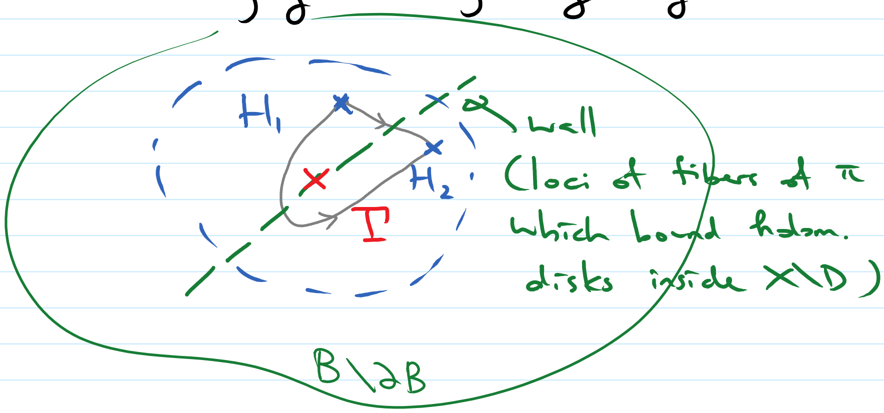




SYZ: deform $\check{J}_0$ by instanton corrections coming from holom. disks in X bounded by fibers of π; in turn, these holom. disks glue to give holom. spheres in X giving rise to genus 0 GW invariants of X.



To construct the mirror mfd $\check{X}$, we can use a trick due to Auroux: instead of deforming $\check{J}_0$, we construct $\check{X}$ by modifying the gluing by wall-crossing data



$W = \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta)=2}} n_\beta \cdot z_\beta$ is different over H_1 and H_2

$\longrightarrow$ W is a multi-valued function.

Auroux's Idea: Floer theory tells us exactly how to correct the mirror so that W becomes single-valued.

$\Rightarrow$ The right correction is given by requiring W to be a single-valued function.

e.g. ① (Auroux 2007)

$$X = \underset{(x,y)}{\overset{\psi}{\mathbb{C}^2}}, \quad D = \{xy = 1\} = \{f = 1\}, \quad \Omega = dx \wedge dy$$

$$\omega = \omega_{std} = -\frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$$

To construct a Lagrangian torus fibration on X, consider the fcn

$$f : X \to \mathbb{C}$$
$$(x,y) \mapsto xy$$

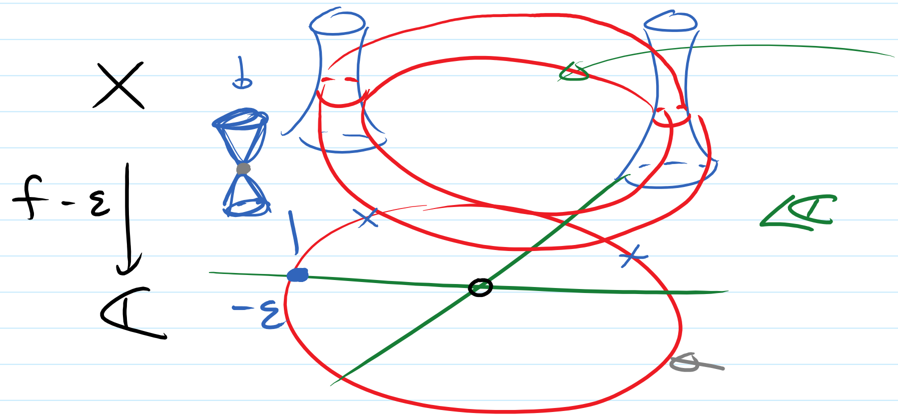
• the $S^1 \curvearrowright X$: $e^{i\theta}\cdot(x,y) = (e^{i\theta}x, e^{-i\theta}y)$

Then the map

$$\begin{array}{ccccc} D & \subset & X & & (x,y) \\ \downarrow & & \downarrow \pi & & \downarrow \\ \mathbb{R}\times\{0\} = \partial B & \subset & B := \mathbb{R}\times\mathbb{R}_{\geq 0} & & \left(\mu_{S^1} = \tfrac{1}{2}(|x|^2 - |y|^2),\ \underbrace{|f(x,y)-1|}_{r}\right) \end{array}$$

π is a Lagrangian torus fibration :

a fiber of π is given as follows



fiber of $f-\varepsilon$ over $a \in \mathbb{C}$

$= \{xy - 1 = a\}$

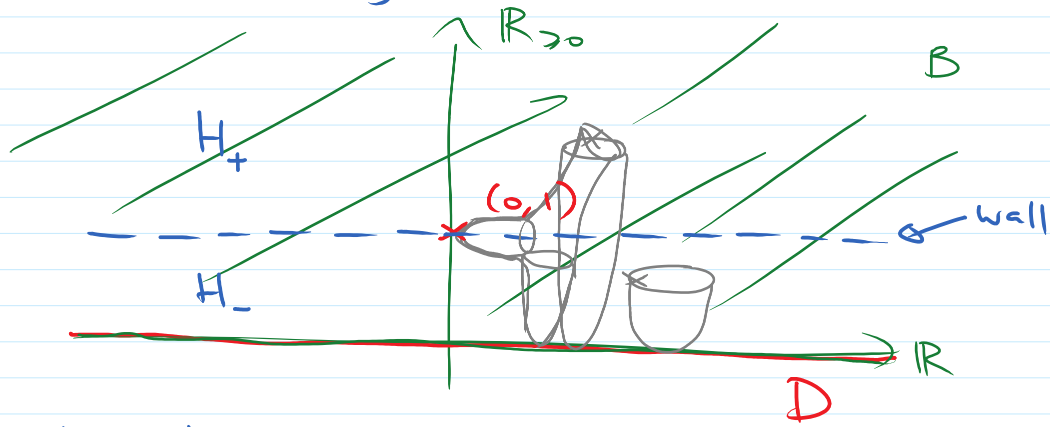
$= \{xy = 1 + a\}$

sing $\iff$ $a = -1$

Furthermore, a fiber becomes singular only when it hits a singular fiber of $f-\varepsilon$, and $\mu_{S^1} = 0$.

This happens only when $|f-1| =$ and $\mu_{S^1} = 0$.

$\implies$ the only singular pt in B is $(0, 1)$



It can be shown that

$$W = \begin{cases} W_+ = y + yw & \text{over } H_+ \end{cases}$$

$$W = \begin{cases} W_+ = y + yw & \text{over } H_+ \\ W_- = u & \text{over } H_- \end{cases}$$

This tells us how to correct the mirror, namely, letting $v = y^{-1}$, we have

$$\check{X}_{syz} := \{ uv = 1 + w \} \subset \mathbb{C}^2 \times \mathbb{C}^\times$$

$$\Updownarrow$$

$$W_+ = W_-$$